Mathematical Systems Theory

third edition

G.J. Olsder
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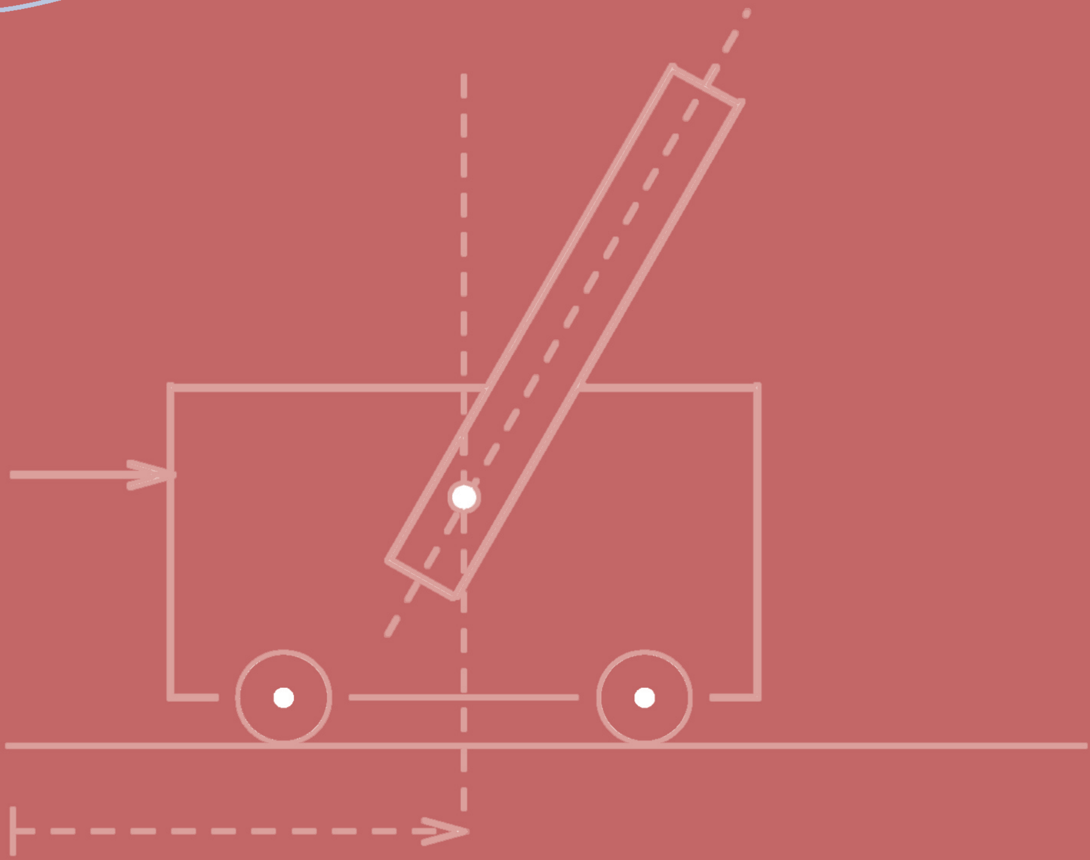

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Preface

Third edition

Compared to the second edition, the presentation of material in this third edition has been changed significantly. For a start, based on feedback by students, certain topics, like linearization, Routh's criterion, interval stability, observer and compensator design, have been discussed in some more detail than in the second edition. Further, in each chapter theorems, lemmas, examples, and so on, are numbered consecutively now, and exercises have been moved towards the end of chapters. Also additional exercises have been included. Finally, errors and typos, found in the second edition, have been corrected. A.A. Stoorvogel and J.G. Maks are greatly acknowledged for their remarks on the second edition. We also thank VSSD for its willingness to publish these notes as a book. We hope that this third edition will be as successful as the previous ones.

Delft, November 2004 G.J. Olsder and J.W. van der Woude

Second edition

The main changes of this second edition over the first one are (i) the addition of a chapter with MATLAB®[1] exercises and possible solutions, and (ii) the chapter on 'Polynomial representations' in the first edition has been left out. A summary of that chapter now appears as a section in chapter 8. The material within the chapter on 'Input/output representations' has been shifted somewhat such that the parts dealing with frequency methods form one section now. Moreover, some exercises have been added and some mistakes have been corrected. I hope that this revised edition will find its way as its predecessor did.

Delft, December 1997 G.J. Olsder

First edition

These course notes are intended for use at undergraduate level. They are a substantial revision of the course notes used during the academic years 1983-'84 till 1993-'94. The most

[1] MATLAB is a registered trademark of The MathWorks, Inc.

notable changes are an omission of some abstract system formulations and the addition of new chapters on modelling principles and on polynomial representation of systems. Also changes and additions in the already existing chapters have been made. The main purpose of the revision has been to make the student familiar with some recently developed concepts (such as 'disturbance rejection') and to give a more complete overview of the field.

A dilemma for any author of course notes, of which the total contents is limited by the number of teaching hours and the level of the students (and of the author!), is what to include and what not. One extreme choice is to treat a few subjects in depth and not to talk about the other subjects at all. The other extreme is to touch upon all subjects only very briefly. The choice made here is to teach the so-called state space approach in reasonable depth (with theorems and proofs) and to deal with the other approaches more briefly (in general no proofs) and to provide links of these other approaches with the state space approach.

The most essential prerequisites are a working knowledge of matrix manipulations and an elementary knowledge of differential equations. The mathematics student will probably experience these notes as a blend of techniques studied in other (first and second year) courses and as a solid introduction to a new field, viz. that of mathematical system theory, which opens vistas to various fields of application. The text is also of interest to the engineering student, who will, with his background in applications, probably experience these notes as more fundamental. Exercises are interspersed throughout the text; the student should not skip them. Unlike many mathematics texts, these notes contain more exercises (61) than definitions (31) and more examples (56) that theorems (36).

For the preparation of these notes various sources have been consulted. For the first edition such a source was, apart from some of the books mentioned in the bibliography, 'Inleiding wiskundige systeemtheorie' by A.J. van der Schaft, Twente University of Technology. For the preparation of these revised notes, also use was made of 'Course d'Automatique, Commande Linéaire des Systèmes Dynamiques' by B. d'Andréa-Novel and M. Cohen de Lara, Ecole Nationale Supérieure des Mines de Paris. The contents of Chapter 2 have been prepared by J.W. van der Woude, which is gratefully acknowledged. The author is also grateful to many of his colleagues with whom he had discussions about the contents and who sometimes proposed changes. The figures have been prepared by Mrs T. Tijanova, who also helped with some aspects of the LaTeX document preparation system by means of which these notes have been prepared.

Parallel to this course there are computer lab sessions, based on MATLAB, by means of which the student himself can play with various examples such as to get a better feeling for concepts and for designing systems himself. This lab has been prepared by P. Twaalfhoven and J.G. Braker.

Delft, April 1994 G.J. Olsder

Contents

Chapter 1

Introduction

1.1 What is mathematical systems theory?

A system is part of reality which we think to be a separated unit within this reality. The reality outside the system is called the surroundings. The interaction between system and surroundings is realized via quantities, quite often functions of time, which are called input and output. The system is influenced via the input(-functions) and the system itself has an influence on the surroundings by means of the output(-functions).

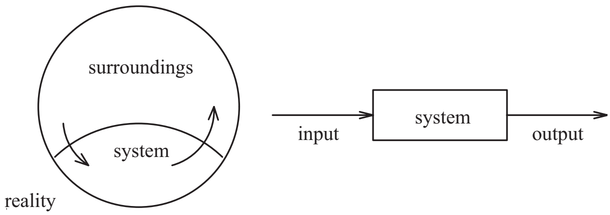

Figure 1.1 A system in interaction with its environment.

Three examples:

- How to fly an aeroplane: the position of the control wheel (the input) has an influence on the course (the output).
- In economics: the interest rate (the input) has an influence on investment behavior (the output).
- Rainfall (the input) has an influence on the level of the water in a river (the output).

In many fields of study, a phenomenon is not studied directly but indirectly through a model of the phenomenon. A model is a representation, often in mathematical terms, of what are felt to be the important features of the object or system under study. By the manipulation of the representation, it is hoped that new knowledge about the modelled phenomenon can be obtained without the danger, cost, or inconvenience of manipulating the real phenomenon itself. In mathematical system theory we only work with models and when talking about a system we mean a modelled version of the system as part of reality.

Most modelling uses mathematics. The important features of many physical phenomena can be described numerically and the relations between these features can be described by equations or inequalities. Particularly in natural sciences and engineering, properties such as mass, acceleration and forces can be described in mathematical terms.

To successfully utilize the modelling approach, however, knowledge is required of both the modelled phenomena and properties of the modelling technique. The development of high-speed computers has greatly increased the use and usefulness of modelling. By representing a system as a mathematical model, converting it into instructions for a computer and running the computer, it is possible to model larger and more complex systems than ever before.

Mathematical system(s) theory is concerned with the study and control of input/output phenomena. There is no difference between the terminologies ‘system theory’ and ‘systems theory’; both are used in the (scientific) literature and will be used interchangeably. The emphasis in system(s) theory is on the dynamic behavior of these phenomena, i.e., how do characteristic features (such as input and output) change in time and what are the relationships between them, also as functions of time. One tries to design control systems such that a desired behavior is achieved. In this sense mathematical system(s) theory (and control theory) distinguishes itself from many other branches of mathematics by the fact that is prescriptive rather than descriptive.

Mathematical system theory forms the mathematical base for technical areas such as automatic control and networks. It is also the starting point for other mathematical subjects such as optimal control theory and filter theory. In optimal control theory one tries to find an input function which yields an output function that satisfies a certain requirement as well as possible. In filter theory the input to the system, then being a so-called filter, consists of observations with measurement errors, while the system itself tries to realize an output which equals the ‘ideal’ observations, i.e., as much as possible without measurement errors. Mathematical system theory also plays a role in economics (specially in macro-economic control theory and time series analysis), theoretical computer science (via automaton theory and Petri-nets) and management science (models of firms and other organizations). Lastly, mathematical system theory forms the hard, mathematical, core of more philosophically oriented areas such as general systems theory and cybernetics.

Example 1.1 [Autopilot of a boat] An autopilot is a device which receives as input the heading $\alpha(t)$ of a boat at time t (measured by an instrument such as a magnetic compass or a gyrocompass) and the (fixed) desired heading α_c (reference point) by the navigator. Using this information, the device automatically yields, as a function of time t, the positioning command $u(t)$ of the rudder in order to achieve the smallest possible heading error $e(t) = \alpha_c - \alpha(t)$. Given the dynamics of the boat and the external perturbations (wind, swell, etc.) the theory of automatic control helps to determine a control input command $u = f(e)$ that meets the imposed technical specifications (stability, accuracy, response time, etc.). For example, this control might be bang-bang:

$$u(t) = \begin{cases} +u_{\max} & \text{if} \quad e(t) > 0, \\ -u_{\max} & \text{if} \quad e(t) < 0. \end{cases}$$

(The arrows in the left-hand picture in Figure 1.2 point in the positive direction of the quantities concerned.) Alternatively, the control might be proportional:

$$u(t) = Ke(t),$$

where K is a constant. It has tacitly been assumed here that for all e-values of interest, $-u_{\max} \leq Ke(t) \leq u_{\max}$. If this is not the case, some kind of saturation must be intro-

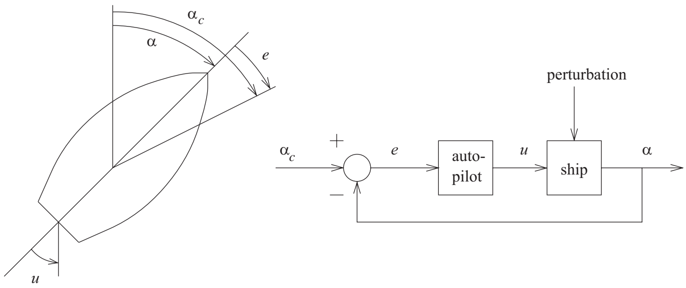

Figure 1.2 Autopilot of a boat.

duced. The control law might also consist of a proportional part, an integrating part and a differentiating part:

$$u(t) = Ke(t) + K'\int^{t} e(s)ds + K''\frac{d}{dt}e(t), \tag{1.1}$$

where K, K' and K'' are constants. This control law is sometimes referred to as a PID controller, where P stands for the proportional part, I for the integrating part and D for the differentiating part. The lower bound of the integral in (1.1) has not been given explicitly; various choices are possible. In all these examples of a control law, a signal (the error in this case) is fed back to the input. One speaks of control by feedback.

Automatic control theory helps in the choice of the best control law. If the ship itself is considered as a system, then the input to the ship is the rudder setting u (and possibly perturbations) and the output is the course α. The autopilot is another system. Its input is the error signal e and output is the rudder setting u. Thus, we see that the output of one system can be the input of another system. The combination of ship, autopilot and the connection from α to α_c, all depicted in the right-hand side of Figure 1.2, can also be considered as a system. The inputs of the combined system are the desired course α_c and possible perturbations, and the output is the real course α. □

Example 1.2 [Optimal control problem] The motion of a ship is described by the differential equation

$$\dot{x} = f(x,u,t),$$

where the state vector $x = (x_1, x_2)^\top \in \mathbb{R}^2$ equals the ship's position with respect to a fixed reference frame, the vector $u = (u_1, u_2)^\top \in \mathbb{R}^2$ denotes the control and t is the time. The superscript $^\top$ refers to 'transposed'. If not explicitly stated differently, vectors are always supposed to be column vectors. Although not specifically indicated, both x and u are supposed to be functions of time. The notation $\dot{x}$ refers to the time derivative of the (two) state components. In this example one control variable to be chosen is the ship's heading u_1, whereas the other one, u_2, is the ship's velocity. The problem now is to choose u_1 and u_2 in such a way that the ship uses as little fuel as possible such that, if it leaves Rotterdam at a certain time, it arrives in New York not more than 10 days later. The functions u_1 and u_2 may depend on available information such as time, weather forecast, ocean streams, and so on. Formally, $u = (u_1, u_2)^\top$ must be chosen such that

$$\int_{t_0}^{t_f} g(x,u,t)dt$$

is minimized, where the (integral) criterion describes the amount of fuel used. The function g is the fuel consumption per time unit, t_0 is the departure time and t_f is the arrival time. □

Example 1.3 [Filtering] NAVSAT is the acronym for NAVigation by means of SATellites. It refers to a worldwide navigation system studied by the European Space Agency (ESA). During the 1980s the NAVSAT system was in the development phase with feasibility studies being performed by several European aerospace research institutes. At the National Aerospace Laboratory (NLR), Amsterdam, the Netherlands, for instance, a simulation tool was developed by which various alternative NAVSAT concepts and scenarios could be evaluated.

Recently, the United States and the European Union have reached an agreement on sharing their satellite navigation services, i.e., the current U.S. Global Positioning System and Europe's Galileo system, which is scheduled to be in operation by 2008. NAVSAT can be seen as a forerunner of Galileo.

The central idea of satellite based navigation system is the following. A user (such as an airplane, a ship or a car) receives messages from satellites, from which he can estimate his own position. Each satellite broadcasts its own coordinates (in some known reference frame) and the time instant at which this message is broadcast. The user measures the time instant at which he receives this message on his own clock. Thus, he knows the time difference between sending and receiving the message, which yields the distance between the position of the satellite and the user. If the user can calculate these distances with respect to at least three different satellites, he can in principle calculate his own position. Complicating factors in these calculations are: (i) different satellites send messages at different time instants, while the user moves in the meantime, (ii) several different sources of error present in the data, e.g. unknown ionospheric and tropospheric delays, the clocks of the satellites and of the user not running exactly synchronously, the satellite position being broadcast with only limited accuracy.

The problem to be solved by the user is how to calculate his position as accurately as possible, when he gets the information from the satellites and if he knows the stochastic characteristics of the errors or uncertainties mentioned above. As the satellites broadcast the information periodically, the user can update also periodically the estimate of his position, which is a function of time. □

1.2 A brief history

Feedback - the key concept of system theory - is found in many places such as in nature and in living organisms. An example is the control of the body temperature. Also, social and economic processes are controlled by feedback mechanisms. In most technical equipment use is made of control mechanisms.

In ancient times feedback was already applied in for instance the Babylonic water wheels and for the control of water levels in Roman aqueducts. According to historian

Otto Mayr, the first explicit use of a feedback mechanism has been designed by Cornelis Drebbel [1572–1633], both an engineer and an alchemist. He designed the ‘Athanor’, an oven in which he optimistically hoped to change lead into gold. Control of the temperature in this oven was rather complex and the method invented by Drebbel could be viewed as a feedback design.

Drebbel’s invention was then used for commercial purposes by his son in law, Augustus Kuffler [1595–1677], being a temporary of Christian Huygens [1629–1695]. It was Christian Huygens who designed a fly-wheel for the control of the rotational speed of windmills. This idea was refined by R. Hooke [1635–1703] and J. Watt [1736–1819], the latter being the inventor of the steam engine. In the middle of the 19th century more than 75,000 James Watt’s fly-ball governors (see Exercise 1.4.2) were in use. Soon it was realized that these contraptions gave problems if control was too rigid. Nowadays one realizes that the undesired behavior was a form of instability due to a high gain in the feedback loop. This problem of bad behavior was investigated J.C. Maxwell [1831–1879] – the Maxwell of the electromagnetism – who was the first to perform a mathematical analysis of stability problems. His paper ‘On Governors’ can be viewed as the first mathematical article devoted to control theory.

The next important development started in the period before the Second World War, in the Bell Labs in the USA. The invention of the electronic amplification by means of feedback started the design and use of feedback controllers in communication devices. In the theoretical area, frequency-domain techniques were developed for the analysis of stability and sensitivity. H. Nyquist [1889–1976] and H.W. Bode [1905–1982] are the most important representatives of this direction.

Norbert Wiener [1894–1964] worked on the fire-control of anti-aircraft defence during the Second World War. He also advocated control theory as some kind of artificial intelligence as an independent discipline which he called ‘Cybernetics’ (this word was already used by A.M. Ampere [1775–1836]).

Mathematical system theory and automatic control, as known nowadays, found their feet in the 1950s. Classic control theory played a stimulating role. Initially mathematical system theory was more or less a collection of concepts and techniques from the theory of differential equations, linear algebra, matrix theory, probability theory, statistics, and, to a lesser extent, complex function theory. Later on (around 1960) system theory got its own face, i.e., ‘own’ results were obtained which were especially related to the ‘structure’ of the ‘box’ between input and output, see the right-hand side picture in Figure 1.1. Two developments contributed to that. Firstly, there were fundamental theoretical developments in the nineteen fifties. Names attached to these developments are R. Bellman (dynamic programming), L.S. Pontryagin (optimal control) and R.E. Kalman (state space models and recursive filtering). Secondly, there was the invention of the chip at the end of the nineteen sixties and the subsequent development of micro-electronics. This led to cheap and fast computers by means of which control algorithms with a high degree of complexity could really be used.

1.3 Brief description of contents

In the present chapter a very superficial overview is given of what system theory is and the relations with other (mainly: technically oriented) fields are discussed. One could say that in this chapter the 'geographical map' is unfolded and that in the subsequent chapters parts of the map are studied in (more) detail.

In Chapter 2 modelling techniques are discussed and as such the chapter, strictly speaking, does not belong to the area of system theory. Since, however, the starting point in system theory always is a model or a class of models, it is important to know about modelling techniques and the principles underlying such models. Such principles are for instance the conservation of mass and of energy. A classification of the variables involved into input (or: control) variables, output (or: measurement) variables, and variables which describe dependencies within the model itself, will become apparent.

In Chapters 3, 4 and 5 the theory around the important class of linear differential systems is dealt with. The reason for studying such systems in detail is twofold. Firstly, many systems in practice can (at least: approximately) be described by linear differential systems. Secondly, the theory for these systems has been well developed and has matured during the last forty years or so. Many concepts can be explained quite naturally for such systems.

The view on systems is characterized by the 'state space approach' and the main mathematical technique used is that of linear algebra. Besides linear algebra one also encounters matrix theory and the theory of differential equations. Chapter 3 deals specifically with linearization and linear differential systems. Chapter 4 deals with structural properties of linear systems. Specially, various forms of stability and relationships between the input, output and state of the system, such as controllability and observability, are dealt with. Chapter 5 considers feedback issues, both state feedback and output feedback, in order to obtain desired system properties. The description of the separation principle is also part of this chapter.

Chapter 6 also deals with linear systems, but now from the input/output point of view. One studies formulas which relate inputs to outputs directly. Main mathematical tools are the theory of the Laplace transform and complex function theory. The advantage of this kind of system view is that systems can easily be viewed as 'blocks' and that one can build larger systems by combining subsystems. A possible disadvantage is that this way of describing systems is essentially limited to linear time-invariant systems, whereas the state space approach of the previous chapters is also suited as a means of describing nonlinear and/or time-dependent systems.

In Chapters 3, 4, 5 and 6 'time' was considered to flow continuously. In Chapter 7 one deals with 'discrete-time' models. Rather than differential equations one now has difference equations which describe the model from the state space point of view. The most crucial concepts of Chapters 4 and 5 are repeated here for such systems. The role of the Laplace transform is taken over by the so-called z-transform. The theories of continuous-time systems and of discrete-time systems are equivalent in many aspects, and therefore Chapter 7 has been kept rather brief. Some modelling pitfalls when approximating a continuous-time system by a discrete-time one are briefly indicated.

Chapter 8 shows some avenues towards related fields. There is an abstract point of view on systems, characterizing them in terms of input space, output space, and maybe state space, and the mappings between these spaces. Also the more recently introduced 'behavioral approach' towards system theory is briefly mentioned. In this approach no distinction is made between inputs and outputs. It is followed by a brief introduction of polynomial matrices used to represent linear systems algebraically. Some remarks on nonlinear systems – a class many times larger than the class of linear systems – will be made together with some progress in this direction. Also other types of systems are mentioned such as descriptor systems, stochastic systems, finite state systems, distributed parameter systems and discrete event systems. Brief introductions to optimal control theory, filter theory, model reduction, and adaptive and robust control will be given. In those fields system theoretical notions introduced earlier are used heavily.

Lastly, Chapter 9 contains a collection of problems and their solutions that can be used for a course on system theory. The problems are solved using the software package MATLAB. For most of them also the MATLAB *Control Toolbox* must be used. The nature of this chapter is clearly different from that of the others.

Books mentioned in the text and some 'classics' in the field of systems theory are given in the bibliography. This book ends with an index.

1.4 Exercises

Exercise 1.4.1 *The water clock ('clepsydra') invented by Ktesibios, a Greek of the third century before Christ, is an old and very well known example of* ***feedback control*** *(i.e., the error is fed back in order to make corrections). Look this up and give a schematic drawing of the water clock with control.*

Exercise 1.4.2 *Another example of an old control mechanism is Watt's centrifugal governor for the control of a steam engine. Consult the literature and find out how this governor works. See for instance [Faurre and Depeyrot, 1977].*

Exercise 1.4.3 *Determine how a float in the water reservoir of a toilet operates.*

Exercise 1.4.4 *Investigate the working of a thermostat in the central heating of a greenhouse. Specify the controls and the measurements.*

Exercise 1.4.5 *Describe how feedback plays a role when riding a bicycle. What are the inputs/controls and what are the outputs/measurements.*

Exercise 1.4.6 *Investigate the mechanism of your body to control its temperature. What is the control action?*